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LETTER TO THE EDITOR

Parasupersymmetric quantum mechanics of arbitrary order

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Abstract. The parasupersymmetric quantum mechanics of one boson and one parafermion of (arbitrary) order p are constructed. The parasupersymmetry algebra is $Q_1^p Q_1^+ + Q_1^{p-1} Q_1^+ Q_1 + \ldots + Q_1 Q_1^+ Q_1^{p-1} + Q_1^+ Q_1^p = 2pQ_1^{p-1} H$; $Q_1^{p+1} = 0$ and $[H, Q_1] = 0$ where Q_1 is the parasupercharge and H is the Hamiltonian. It is also shown that such a system always possesses $(p-1)$ other conserved parasupercharges and p bosonic constants. A model of conformal parasupersymmetry of order p is also discussed and it is shown that in this case the algebra has a particularly simple form.

In the last few years supersymmetric quantum mechanics [1] (SQM) has been extensively discussed in the literature. Because of this one has a better understanding of analytically solvable potentials, the relation between spectra of different Hamiltonians, etc. [2]. More than three years back Rubakov and Spiridonov (RS) [3] extended this discussion to the case of parasupersymmetric quantum mechanics (PSQM) of order two. In particular they show that the algebra is given by

$$Q_1^3 = 0 \qquad [H, Q_1] = 0 \tag{1a}$$

$$Q_1^2 Q_1^+ + Q_1 Q_1^+ Q_1 + Q_1^+ Q_1^2 = 4Q_1 H \tag{1b}$$

and the Hermitian conjugated relations. Various consequences of this algebra have been discussed by these and other authors. This work has now been extended in several directions [4]. For example, a model of conformal PSQM of order two has been discussed in the literature [5]. The obvious question is if one can generalize the PSQM of order two to that of an (arbitrary) order p [6]. This question was raised by RS themselves [3] who were able to write down the bilinear part of PSQM of order p which is essentially an obvious generalization of (1a), i.e.

$$Q_1^{p+1} = 0 \qquad [H, Q_1] = 0. \tag{2}$$

However they could not write the remaining multilinear part of the algebra. As far as the author is aware, this problem has remained unsolved till today even though such multilinear algebra has been written down in the case of special choices of superpotentials [7]. In fact recently Durand *et al* [8] have discussed this issue in some detail and have concluded that the multilinear part of the higher-order PSQM ($p \geqslant 3$) cannot be characterized with one universal algebraic relation.

In this letter a fresh look at this problem is taken and it is shown that the multilinear part of the PSQM of order p can in fact be characterized with one universal algebraic relation. In particular it is shown that the generalization of relation (1b) for arbitrary p is

$$Q_1^p Q_1^+ + Q_1^{p-1} Q_1^+ Q_1 + \ldots + Q_1 Q_1^+ Q_1^{p-1} + Q_1^+ Q_1^p = 2pQ_1^{p-1} H, \ldots \qquad p = 1, 2 \ldots. \tag{3}$$

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Notice that one has $(p+1)$ terms on the LHS of this relation. Not surprisingly, for $p=1$ we recover the algebraic relations of SQM. Some consequences of the PSQM as given by equations (2) and (3) are also discussed. In particular it is shown that the hierarchy of Hamiltonians $H_1, H_2, \ldots, H_p$ which are connected by SQM and which have $p, p-1, \ldots, 1$ bound states, respectively, form PSQM of order p. Further it is shown that PSQM of degree p always possesses $(p-1)$ other conserved supercharges $Q_2, Q_3, \ldots, Q_p$ and (p) bosonic constants. Also discussed is a special case when the PSQM of degree p can be characterized by simpler multilinear relations like

$$Q_1 Q_1^+ Q_1 = 2Q_1 H \tag{4a}$$

$$Q_1^p Q_1^+ + Q_1^+ Q_1^p = 2Q_1^{p-1} H. \tag{4b}$$

Finally, a model for the conformal PSQM of order p is discussed and it is shown that in this case one has very simple algebra, a part of which is as given by equations (4).

To motivate the whole discussion, let us consider parafermionic operators a and a^+ of order p. They satisfy the algebra [6]

$$(a)^{p+1} = 0 = (a^+)^{p+1}, \ldots \qquad p = 1, 2, 3, \ldots \tag{5a}$$

$$[[a^+, a], a] = -2a \qquad [[a^+, a], a^+] = a^+. \tag{5b}$$

On letting

$$J_+ = a^+ \qquad J_- = a \qquad J_3 = \tfrac{1}{2}[a^+, a] \tag{6}$$

it immediately follows from equations (5) and (6) that the operators $J_\pm$, J_3 satisfy the SU(2) commutation relations

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm. \tag{7}$$

Let us choose J_3 to represent the third component of spin-$p/2$ representation of the SU(2) group as given by $J_3 = \mathrm{dig}(p/2, p/2-1, \ldots, -p/2+1, -p/2)$. It is then easily checked that the operators a and a^+ can be represented by the following $(p+1)\times(p+1)$ matrices

$$(a)_{\alpha,\beta} = C_\beta \delta_{\alpha,\beta+1} \qquad (a^+)_{\alpha,\beta} = C_\alpha \delta_{\alpha+1,\beta} \tag{8a}$$

where

$$C_\beta = \sqrt{\beta(p-\beta+1)} = C_{p-\beta+1} \qquad \alpha, \beta = 1, 2, \ldots, p+1. \tag{8b}$$

One can now ask what multilinear relation is satisfied by a and a^+ apart from the bilinear relations as given in (5). It is easily checked that such a multilinear relation is given by $(p = 1, 2, 3, \ldots)$

$$a^p a^+ + a^{p-1} a^+ a + \ldots + a a^+ a^{p-1} + a^+ a^p = \tfrac{1}{6} p(p+1)(p+2) a^{p-1} \tag{9}$$

where on the LHS of the equation one has $p+1$ number of terms. This relation strongly suggests that one may have an analogous multilinear relation in the algebra of PSQM of order p.

Following RS [3] we choose the parasupercharges Q_1 and Q_1^+ as $(p+1)\times(p+1)$ matrices as given by $(\alpha, \beta = 1, 2, \ldots, p+1)$

$$(Q_1)_{\alpha\beta} = (P - \mathrm{i} W_\beta)\delta_{\alpha,\beta+1} \qquad (Q_1^+)_{\alpha\beta} = (P + \mathrm{i} W_\alpha)\delta_{\alpha+1,\beta}. \tag{10}$$

Note that, unlike previous workers, we have chosen opposite signs for W. It is not very difficult to show that the Hamiltonian ($m=1$, $\hbar=1$)

$$H=+\frac{1}{2}\begin{pmatrix} h_1 & 0 & \dots & \dots & 0 \\ 0 & h_2 & & & \\ \vdots & & \ddots & & \\ 0 & 0 & & h_p & 0 \\ 0 & 0 & \dots & 0 & h_{p+1} \end{pmatrix} \tag{11}$$

where

$$h_r = p^2 + W_r^2 - W_r' + C_r \qquad r=1,2,\dots,p \tag{12a}$$

$$h_{p+1} = P^2 + W_p^2 + W_p' + C_p \tag{12b}$$

commutes with the supercharges Q_1 and Q_1^+ provided [8]

$$W_r^2 + W_r' + C_r = W_{r+1}^2 - W_{r+1}' + C_{r+1} \qquad r=1,2,\dots,(p-1). \tag{13}$$

Here $C_1, C_2, \dots, C_p$ are arbitrary constants with dimensions of energy. On using equations (10) to (13) one can show that Q_1, Q_1^+ and H satisfy the highly non-trivial multilinear algebra as given by (3) provided

$$C_1 + C_2 + \dots + C_p = 0. \tag{14}$$

In the special case when all the constants vanish ($C_1 = C_2 = \dots = C_p = 0$) then one can, in fact, show that Q_1 and Q_1^+ as given by (10), satisfy a very simple algebra as given by equations (2) and (4a). On using (2) and (4a) it follows that

$$Q_1^{p-r} Q_1^+ Q_1^r = 2 Q_1^{p-1} H \qquad r=1,2,\dots,p-1 \tag{15}$$

so that relation (4b) immediately follows by making use of (15) in (4a).

Some consequences of PSQM of order p are:

(1) The spectrum is $(p+1)$-fold degenerate at least above the first p levels. The ground state could be $1, 2, \dots, (p+1)$-fold degenerate depending on the form of the superpotentials.

(2) With the Hamiltonian H one can associate p ordinary SQM Hamiltonians. For example

$$H_{\mathrm{SUSY}}^{(r)} = \begin{pmatrix} h_r - c_r & 0 \\ 0 & h_{r+1} - c_r \end{pmatrix}. \tag{16}$$

(3) The PSQM system of order p can describe the motion of a particle with spin $p/2$ in a particular (oscillator or Morse) potential and magnetic field related to this potential.

(4) There is an interesting application of PSQM in the standard quantum mechanics. For example, it is well known that given any potential H_1 with p bound states with energies $E_1, E_2, \dots, E_p$ one can always generate p Hamiltonians $H_1, H_2, \dots, H_p$ which have the same spectrum as H_1 except that $0, 1, \dots, p-1$ levels respectively are missing from them. It is not very difficult to show that the hierarchy of Hamiltonian H_1, $H_2, \dots, H_p$ form a PSQM system of order p and where the constants $C_1, C_2, \dots, C_p$ are given by ($r=1, 2, \dots, p$)

$$C_r = \frac{2}{p}[E_p + E_{p-1} + \dots - (p-1)E_r + E_{r-1} + \dots + E_1]. \tag{17}$$

Some other symmetry groups have also been identified with these hierarchy of Hamiltonians [9].

(5) Another application of PSQM is in the case of strictly isospectral Hamiltonians. For example, it is well known that given a potential $V_1(x)$ with at least one bound state one can always construct one continuous parameter family of potentials given by [10] ($\lambda > 0$ or $\lambda < -1$)

$$V_1(x, \lambda) = V_1(x) - 2\frac{d^2}{dx^2}\ln(I(x)+\lambda) \tag{18a}$$

$$I(x) = \int_{-\infty}^{x} \psi_0^2(y)\,dy \tag{18b}$$

with $\psi_0(x)$ being the normalized ground state wavefunction of $V_1(x)$. It is not difficult to show that the three potentials $V_1(x) \equiv W_1^2 - W_1'(x)$, $V_2(x) = W_1^2 + W_1'(x)$ and $V_3(x) \equiv V_1(x, \lambda)$ form PSQM of order 2 in the case $C_1 = C_2 = 0$.

We now show that given PSQM of order p, apart from Q_1 one has in fact $(p-1)$ other conserved (independent) supercharges $Q_2, Q_3, \ldots, Q_p$ which are defined by

$$(Q_r)_{\alpha\beta} = (P - \mathrm{i}W_\beta)\delta_{\alpha,\beta+1} \qquad \beta \neq r \tag{19a}$$

$$= -(P - \mathrm{i}W_\beta)\delta_{\alpha,\beta+1} \qquad \beta = r \tag{19b}$$

where $\alpha, \beta = 1, 2, \ldots, p+1$, $r = 2, 3, \ldots, p$. It is immediately checked that each of these Q_r, Q_r^+ satisfy the algebra of PSQM of order p given by (2) and (3). What are the analogues of equations (3) when more than one Q_i are involved? To that end notice that apart from $\mathbb{1}$, one also has p bosonic constant I_P defined by

$$(I_i)_{\alpha\beta} = \delta_{\alpha\beta} \qquad \alpha \neq i \tag{20a}$$

$$= -\delta_{\alpha\beta} \qquad \alpha = i \tag{20b}$$

where $i = 2, 3, \ldots, p+1$ and $\alpha, \beta = 1, 2, \ldots, (p+1)$. It is immediately seen that $[H, I_i] = 0$, $[I_i, Q_j] = \Sigma_{K=1}^{p} \ldots d_K Q_K$ with d_K being constants. Using these bosonic constants one can now write down relations analogous to those of (3) involving all the supercharges Q_i. For example, in the case of $p = 2$, one has two supercharges Q_1 and Q_2 and the mixed relations analogous to those of (3) are ($i, j = 1, 2, \quad i \neq j$)

$$Q_j Q_j^+ Q_i + Q_i Q_j Q_j^+ + I_2 Q_j^+ Q_i Q_j = 4 Q_i H \tag{21a}$$

$$I_3 Q_j^2 Q_i^+ + I_2 Q_i^+ Q_j^2 + Q_j Q_i^+ Q_j = 4 Q_i H. \tag{21b}$$

It is worth noting that these relations are all independent of the constants C_i unlike some previous attempts in the literature [11]. Generalization of these relations to arbitrary p is immediate.

Finally we discuss a PSQM model of order p which is in addition conformally invariant and show that, unlike the previous claim [8], the parafermionic charges obey very simple relations. In particular we take

$$W_1 = \frac{\lambda}{x},\ W_2 = \frac{\lambda+1}{x}, \ldots, W_p = \frac{\lambda+p-1}{x} \tag{22}$$

so that equations (14) are satisfied with $C_1 = C_2 = C_3 \ldots = C_p = 0$. Hence, as shown above, the parasupersymmetric algebra takes a very simple form and is given by equations (2) and (4). In this case one can also define the dilatation operator D, the conformal operator K, the hypercharge Y and the superconformal charge S_1 by

$$D=-\frac{1}{4}(xp+px) \qquad K=\frac{1}{2}x^2 \qquad Y=\frac{1}{2}\left(J_3-\lambda-\frac{p+1}{2}\right)$$
$$(S_1)_{\alpha\beta}=-x\delta_{\alpha,\beta+1} \qquad \alpha,\beta=1,2,\ldots,(p+1). \tag{23}$$

The algebra satisfied by Q_1 and S_1 with D, K, Y, H is standard [5]. Once again it turns out that one has in fact $(p-1)$ other conserved supercharges and superconformal charges $Q_2,\ldots,Q_p$ and $S_2,\ldots,S_p$ respectively which are defined analogously as in equations (19) and (20). For example in the case of $p=2$ one has two supercharges Q_1, Q_2 and two superconformal charges S_1 and S_2 and they satisfy

$$S_iQ_j^+Q_j=2S_iH \qquad Q_iS_j^+S_j=2Q_iK$$
$$S_iS_j^+S_j=2S_iK \qquad Q_iQ_j^+Q_j=2Q_iH$$
$$Q_jS_j^+Q_i=2Q_i(D+\mathrm{i}Y) \qquad S_jS_j^+Q_i=2S_i(D+\mathrm{i}Y)$$
$$Q_jQ_j^+S_i=2Q_i(D-\mathrm{i}Y) \qquad S_jS_j^+Q_i=2S_i(D+\mathrm{i}Y) \tag{24}$$

plus many other analogous relations. Generalization to the case of arbitrary p is immediate. Details of this work along with the discussion of various related issues will be published elsewhere [12]. Recently *à la* parastatistics, orthostatistics has also been discussed [13]. We have been able to construct a model of orthosupersymmetric quantum mechanics of order p [14].

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